Pergamon

J. Appl. Maths Mechs, Vol. 57, No. 5, pp. 879–885, 1993

0021–8928/93 $24.00+0.00

0021–8928(93)E0044-W

A DYNAMICAL PROBLEM OF THE THEORY OF ELASTICITY FOR ANGULAR DOMAINS WITH HOMOGENEOUS BOUNDARY CONDITIONS†

KH. B. TOLIPOV

Chelyabinsk

(*Received* 3 *March* 1992)

The solutions of a problem of the theory of elasticity for a wedge with homogeneous boundary conditions are obtained as combinations of McDonald functions. Displacement fields of waves propagating inside the wedge are determined, which have not so far been described in the literature. Starting from a physical argument, the displacement amplitudes are determined for reflected, refracted, and volume waves. These agree with experimental data.

1. FORMULATION OF THE PROBLEM

CONSIDER an unbounded wedge-shaped plate with flat surfaces (Fig. 1). The discussion of wave propagation begins with the following vector differential equation of motion of particles when there are no volume forces

$$(\lambda+\mu)\operatorname{grad}\operatorname{div}U+\mu\Delta u=\rho_0\ddot{U} \tag{1.1}$$

where λ and μ are the Lamé coefficients, U is the displacement vector, and ρ_0 is the density of the medium.

We will express the displacement vector in the form [1]

$$U=\operatorname{grad}\varphi+\operatorname{rot}\psi_1 k+\operatorname{rot}\operatorname{rot}\psi_2 k \tag{1.2}$$

where the vector potential is expressed in terms of two scalar functions ψ_1 and ψ_2, and where φ is the scalar potential and k denotes the unit vector along the z-axis. In this case one can succeed in separating the solution of the vector wave equation for each function ψ_1 and ψ_2

$$\Delta\varphi+K_l^2\varphi=0,\ \ \Delta\psi_j+K_t^2\psi_j=0;\ \ j=1,2 \tag{1.3}$$

where K_l and K_t are the wave numbers of longitudinal and transverse waves, respectively.

The boundary conditions have the form

$$\frac{\sigma_{\theta\rho}}{\mu}=\frac{2}{\rho}\frac{\partial^2\varphi}{\partial\rho\partial\theta}-\frac{2}{\rho^2}\frac{\partial\varphi}{\partial\theta}+\frac{1}{\rho}\frac{\partial\psi_1}{\partial\rho}-\frac{\partial^2\psi_1}{\partial\rho^2}+\frac{1}{\rho^2}\frac{\partial^2\psi_1}{\partial\theta^2}+\frac{2}{\rho}\frac{\partial^3\psi_2}{\partial\rho\partial\theta\partial z}-\frac{2}{\rho^2}\frac{\partial^3\psi_2}{\partial\theta^2\partial z}$$

$$\frac{\sigma_{\theta z}}{\mu}=\frac{2}{\rho}\frac{\partial^2\varphi}{\partial\theta\partial z}-\frac{\partial^3\psi_1}{\partial\rho\partial z}-\frac{1}{\rho}\frac{\partial^3\psi_1}{\partial z^2\partial\theta}-\frac{1}{\rho}\frac{\partial^2\psi_2}{\partial\rho\partial\theta}-\frac{1}{\rho^2}\frac{\partial^3\psi_2}{\partial\theta^3}-\frac{\partial^3\psi_2}{\partial\theta\partial\rho^2} \tag{1.4}$$

†*Prikl. Mat. Mekh.* Vol. 57, No. 5, pp. 120–126, 1993.

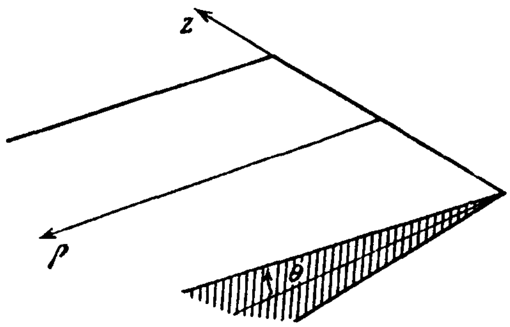

FIG. 1.

$$\frac{\sigma_{\theta\theta}}{2\mu}=-\frac{\partial^2\varphi}{\partial z^2}-\frac{K_t^2}{2}\varphi-\frac{\partial^2\varphi}{\partial\rho^2}+\frac{1}{\rho^2}\frac{\partial^2\psi_1}{\partial\theta^2}-\frac{1}{\rho}\frac{\partial^2\psi_1}{\partial\theta\partial\rho}+\frac{1}{\rho}\frac{\partial^2\psi_2}{\partial\rho\partial z}+\frac{1}{\rho^2}\frac{\partial^3\psi_2}{\partial\theta^2\partial z}.$$

The conditions of the problem require that the stresses $\sigma_{\theta\rho}$, $\sigma_{\theta z}$, $\sigma_{\theta\theta}$ should vanish for the wedge angles $\Theta=\Theta_0$ and $\Theta=-\Theta_0$.

2. PARTICULAR SOLUTION OF THE PROBLEM

Consider the functions

$$\begin{aligned}\varphi&=(a_0\operatorname{ch}\nu_1\theta+b_0\operatorname{sh}\nu_1\theta)\exp[i(pz-\omega t)]\\ \psi_j&=(a_j\operatorname{ch}\nu_2\theta+b_j\operatorname{sh}\nu_2\theta)\exp[i(pz-\omega t)]\end{aligned}\tag{2.1}$$

as solutions of (1.3). Here p is the wave number, ν_1 and ν_2 are the field characteristics connected with the wave numbers, and ω is the angular frequency. Substituting these functions into (1.3), we obtain

$$\begin{aligned}&a_0=A_0K_{\nu_1}(\rho\alpha)+B_0I_{\nu_1}(\rho\alpha),\quad b_0=C_0K_{\nu_1}(\rho\alpha)+D_0I_{\nu_1}(\rho\alpha)\\ &a_j=A_jK_{\nu_2}(\rho\beta)+B_jI_{\nu_2}(\rho\alpha),\quad b_j=C_jK_{\nu_2}(\rho\beta)+D_jI_{\nu_2}(\rho\beta)\\ &\alpha=\sqrt{p^2-K_l^2},\quad \beta=\sqrt{p^2-K_t^2}\end{aligned}\tag{2.2}$$

where I_{ν_i} and K_{ν_i} are the modified Bessel functions of the first and second kind. Using the asymptotic forms of cylindrical functions [2] and the radiation condition

$$K_\alpha(s)\sim\sqrt{\frac{\pi}{2s}}\exp(-s),\quad I_\alpha(s)\sim\frac{1}{\sqrt{2\pi S}}\exp(s),\ \text{при}\ s\to\infty,\quad |\arg s|<\frac{\pi}{2}$$

it can be shown that $B_jD_0\equiv D_j\equiv 0$.

Substituting (2.1) into the boundary conditions (1.4) and equating the stresses to zero, we obtain a system of six equations with unknowns A_0, A_1, A_2, B_0, B_1, B_2. The solution of this system enables us to determine the types of waves propagating in a wedge-shaped plate. The non-zero solution of this system is of interest. A condition for the existence of such a solution is that the determinant of the system should be equal to zero.

Consider plane waves independent of z by virtue of symmetry. The sixth-order determinant can be reduced to the product of three second-order determinants corresponding to three independent families of displacement components.

Solution 1

$$A_0,\ C_0,\ A_1,\ C_1=0,\ \ A_2,\ C_2\neq 0,\ \ U_\rho=U_\theta=0$$

$$U_z=[(\nu_2^2/\rho-1/\rho)K_{\nu_2}(iK_t\rho)-K'_{\nu_2}(iK_t\rho)][A_2\operatorname{ch}\nu_2\theta+C_2\operatorname{sh}\nu_2\theta]$$

Solution 2

$$A_1,\ C_0,\ A_2,\ C_2=0,\ \ A_0,\ C_1\neq 0,\ \ U_z=0$$

$$U_\theta=A_0\rho^{-1}\nu_1K_{\nu_1}(iK_l\rho)\operatorname{sh}\nu_1\theta-C_1K'_{\nu_2}(iK_t\rho)\operatorname{sh}\nu_2\theta$$

$$U_\rho=A_0K'_{\nu_1}(iK_l\rho)\operatorname{ch}\nu_1\theta+C_1\rho^{-1}\nu_2K_{\nu_2}(iK_t\rho)\operatorname{ch}\nu_2\theta$$

Solution 3

$$A_0,\ C_1,\ A_2,\ C_2=0,\ \ A_1,\ C_0\neq 0,\ \ U_z=0$$

$$U_\rho=A_1\rho^{-1}\nu_2K_{\nu_2}(iK_t\rho)\operatorname{sh}\nu_2\theta+C_0K^1_{\nu_1}(iK_l\rho)\operatorname{sh}\nu_1\theta$$

$$U_\theta=-A_1K'_{\nu_2}(iK_t\rho)\operatorname{ch}\nu_2\theta+C_0\rho^{-1}\nu_1K_{\nu_1}(iK_l\rho)\operatorname{ch}\nu_1\theta$$

(the derivative with respect to ρ is denoted by a prime).

In Solution 1 the displacements of particles are perpendicular to the direction of wave propagation. Therefore, the solution corresponds to transverse waves, in which the displacements of the particles of the medium are parallel to the boundary surfaces. The wave motions of particles corresponding to the other two solutions have a complex form. In these solutions two displacement components are non-zero and the superposition of wave motions involves a combination of longitudinal and transverse motions. In Solution 2 the particle displacement vector is symmetric with respect to $\Theta=0$, i.e. with respect to the middle plane passing through the bisectrix of the wedge angle.

In Solution 3 the displacement vector is antisymmetric with respect to the middle plane. We will consider the equations of symmetric and antisymmetric waves, which can be obtained by equating the two corresponding determinants to zero [3, 4]

$$\frac{K^+_{\nu_1}-2(K_t^2/K_l^2-1)K_{\nu_1}(iK_l\rho)}{K^-_{\nu_1}K^-_{\nu_2}}K^+_{\nu_2}-\left(\frac{\operatorname{th}\nu_1\theta}{\operatorname{th}\nu_2\theta}\right)^\kappa=0$$
$$K^\pm_{\nu_j}=K_{\nu_j-2}(iK_l\rho)\pm K_{\nu_j+2}(iK_l\rho) \tag{2.3}$$

($\kappa=1$ for symmetric waves, and $\kappa=-1$ for antisymmetric ones). We express the McDonald functions in these equations in terms of Hankel functions of the first and second kind [2]. To estimate the resulting equations, we apply the Langer formula [2], which, for any $\nu\gg 1$, give asymptotic representations uniform over the interval $0<x<\infty$

$$H^{(1)}_\nu(x)=\sqrt{\lambda}e^{-\frac{2}{3}\pi i}\ \ H^{(2)}_{1/3}(e^{\frac{i}{2}\pi}\nu\omega\lambda)$$

$$(\omega=\sqrt{1-x^2/\nu^2},\ \lambda=\omega^{-1}Ar\operatorname{th}\omega-1)$$

The dependence of the velocity and damping on the distance to the edge of the wedge for various wedge angles obtained from the solutions of (2.3) is presented in Figs 2 and 3. Curves 1–6 correspond to the angles 0.039, 0.05, 0.1, 0.1, 0.05, 0.039, C is the wave velocity in the wedge-shaped plate, and C_R is the Rayleigh wave velocity.

Analysis indicates that the curves characterize waves which have not been described so far. The comparison of these waves with those propagating in a plane-parallel plate reveals the following.

In a wedge-shaped plate, as opposed to a plane-parallel one, there are two types of propagating waves in which the particles move in a plane perpendicular to the surface. For one of these types, the particles move symmetrically relative to the plane passing through the bisectrix of the angle of the wedge, while, for the other type, they move antisymmetrically relative to the plane.

If the velocity of a supersonic wave in a plane-parallel plate is constant for a given thickness, it varies monotonically in a wedge-shaped plate, namely, the velocity of one of the waves increases towards the edge up to a certain value, while the velocity of the other wave decreases to zero. Away from the edge the velocities of both wave types tend to the velocity of a Rayleigh wave.

The nature of wave propagation in a wedge-shaped plate differs from that in a plane-parallel plate. Waves in which the displacements of particles are antisymmetric relative to the plane of symmetry of the wedge reach the edge of the wedge, while waves of the other type do not reach the edge. The length of this "plug" depends on the wavelength λ

$$l = 0.76\lambda \operatorname{ctg}\theta \tag{2.4}$$

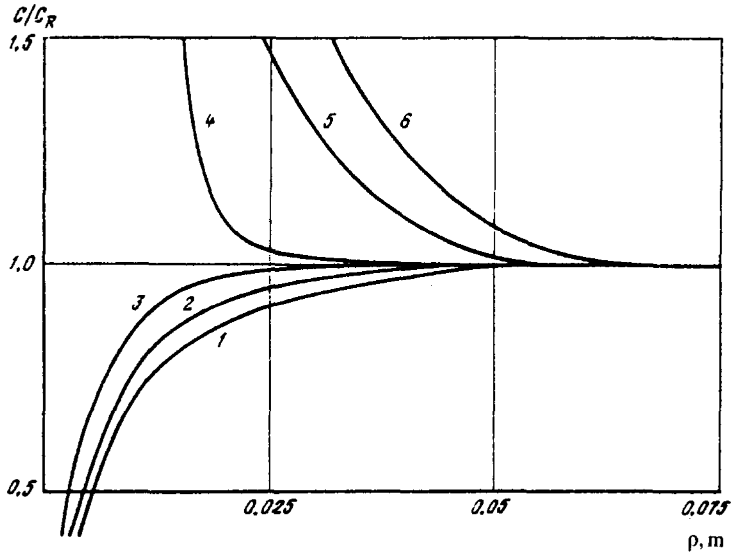

FIG. 2.

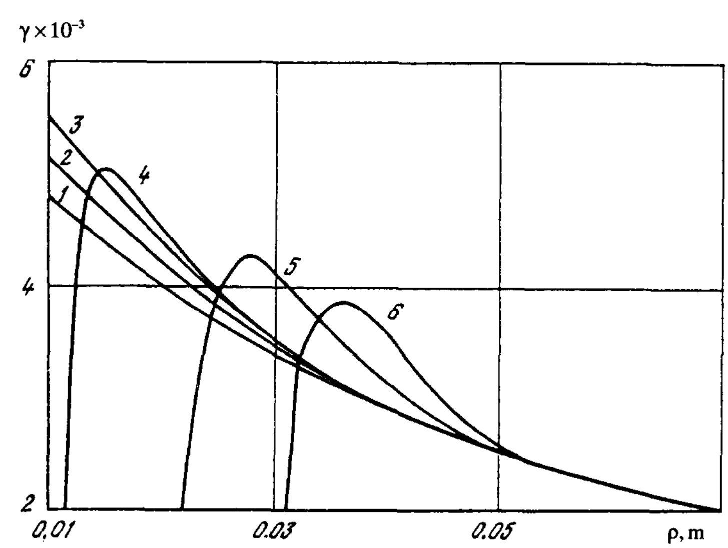

FIG. 3.

3. THE COMPLETE SOLUTION OF THE PROBLEM

The wave field in a wedge consists of reflected surface waves, refracted surface waves, and volume waves. For the amplitudes of such waves to be determined, an additional restriction is necessary, such as the following condition on the edge, which represents the law of conservation of energy [5]

$$U = C + O(r^{\varepsilon})$$

where r is the distance between the current point and the edge of the wedge, and ε is a positive number.

The condition turns out to be too weak to determine the three kinds of waves. We therefore determine the amplitudes from a physical argument followed by the verification of the law of conservation of energy.

A Rayleigh wave propagates far away from the edge which is a set of compression–expansion waves and translation waves characterized by the wave vectors K_l and K_t. As the wave approaches the edge, it splits into two waves, the velocities of which vary in different ways (Fig. 2). The phase shift between the wave vectors of these waves arising in the vicinity of the edge determines the amplitudes of the reflected and refracted surface waves as well as the volume wave. The volume wave is formed as one moves away from the edge and is caused by the increasing velocity of the reflected wave near the surface of the wedge. Moreover, the wave front tears apart and breaks away from the surface. A similar effect occurs when a Rayleigh wave crosses a curved surface. For the waves in question, Fig. 4 shows the dependence of the phase shifts between K_l and K_t on the angle of the wedge. Functions 1, 2, and 3 correspond to the refracted, reflected, and volume waves.

For the refracted surface wave, the phase shift between the wave vectors K_l and K_t appears when the angle of the wedge reaches 175°. This is connected with the fact that, in a Rayleigh wave, K_t is directed at an imaginary angle to the surface of the wedge, and, starting from an angle of incidence of 85°, the angle between K_t and the other face of the wedge is twice as large. The wave vector of the longitudinal wave has variable direction and is connected with the peculiar nature of the propagation of this wave inside the wedge. The "plug" near the edge of the wedge displaces the particles of the medium only in one direction, namely, in the direction of wave motion. As the angle of the wedge decreases, the length of the plug increases, and, for certain angles, the direction of K_l reverses. These angles can be determined from (2.4): $\mathrm{tg}\theta = 0.76K$, where $K = 0.5, 1.0, 1.5, 2.0$. It follows that $\varphi = 2\theta = 133°, 74°, 54°, 42°$.

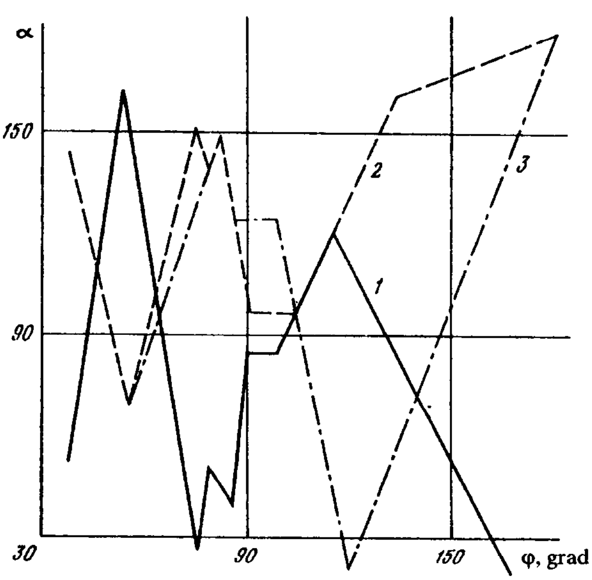

FIG. 4.

For these wedge angles, the phase shift between K_l and K_t undergoes jump-like changes. For wedge angles between 175° and 74°, the phase shift is due to the rotation of K_t and is determined by the principles of geometrical acoustics. For smaller wedge angles, the domain of variation of the wave velocities increases (Fig. 2) and the phase shift between K_l and K_t is essentially determined by the difference between the velocities of the longitudinal and transverse waves. For wedge angles between 103° and 93°, the geometric rotation of K_t is insignificant and comparable with the rotation of this vector connected with the increase in the velocity of the transverse wave. Since these shifts compensate one another, the difference between the phases of K_t and K_l is constant in the interval in question. As the angle of the wedge decreases further, the variation of the phase of K_t due to geometric rotation near 90° first decreases, and then increases to become greater than the phase shift caused by the decreasing velocity of the surface wave. This gives rise to phase shift oscillations in the range of wedge angles between 96° and 85°.

For wedge angles between 180° and 135° the phase shift between K_l and K_t is caused by the difference between the velocities of the longitudinal and transverse waves. Since the angle of incidence of the transverse wave on the other face of the wedge exceeds 45°, the wave is totally reflected. For wedge angles smaller than 135°, there is an additional change in the direction of K_t due to geometric reflection. As the angle of the wedge decreases further below 90°, the variation of the phase shift is caused essentially by the increasing velocity of the transverse wave. The oscillations of the phase shift and the constant phase shift at certain wedge angles are due to the same reasons as in the case of the refracted wave.

The variation of the phase shift between the wave vectors K_l and K_t, as the volume wave is formed, is related to the variation of the phase shift for the reflected wave.

For sufficiently small wedge angles the difference between the velocities of longitudinal and transverse waves increases, which causes the period of phase shift oscillations to decrease.

In Fig. 5 we show curves of the amplitudes of the reflected, refracted, and volume waves against the angle of the wedge. In Fig. 5 we also present experimental values of the amplitudes of the reflected waves (the solid circles) and refracted waves (the hollow circles) obtained for duralumin samples [6].

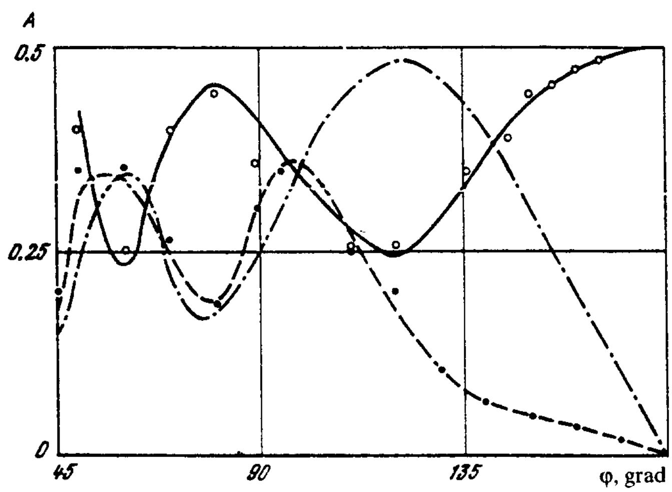

FIG. 5.

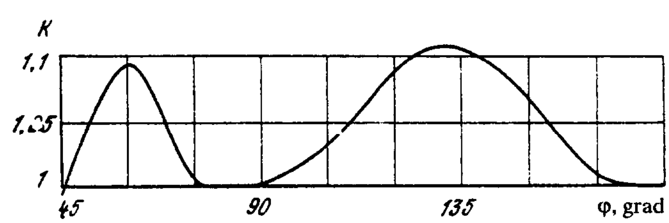

FIG. 6.

In Fig. 6 we present the results of a numerical computation of the function K equal to the sum of the squares of the amplitudes of the reflected, refracted, and volume waves for an incident Rayleigh wave of normalized amplitude. The fact that the values of the function (which are close to one) are in good agreement with the normalized amplitude of the Rayleigh wave indicates that the law of conservation of energy is satisfied.

REFERENCES

1. MORSE P. M. and FESHBACH H., *Methods of Theoretical Physics*, Vol. 2. McGraw-Hill, New York, 1953.
2. YANKE Ye., EMDE F. and LÖSCH F., *Special Functions: Formulae, Graphs, and Tables*. Nauka, Moscow, 1977.
3. TOLIPOV Kh. B., A study of the acoustic field parameters arising in wedge-shaped bodies. *Proceedings of the 5th Conference on Science and Technology*. Tsentr. Dom. Nauchno-Tekh. Prosv., Izhevsk, 1984.
4. TOLIPOV Kh. B., Computation of displacements in a wedge for EMA-generated elastic oscillations. *Collected Papers* (Edited by Yu. V. Petrov). Chelyabinsk, 1988.
5. DOBRUSHKIN V. A., *Boundary Value Problems of the Dynamical Theory of Elasticity for Wedge-shaped Domains*. Nauka i Tekhnika, Minsk, 1988.
6. VITOROV I. A., *Physical Principles of the Applications of Rayleigh and Lamb Ultrasonic Waves in Technology*. Nauka, Moscow, 1966.

Translated by T.J.Z.